

Spontaneous Scalarization and Boson Stars

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Abstract

We study spontaneous scalarization in Scalar-Tensor boson stars. We find that scalarization does not occur in stars whose bosons have no self-interaction. We introduce a quartic self-interaction term into the boson Lagrangian and show that when this term is large, scalarization does occur. Strong self-interaction leads to a large value of the compactness (or sensitivity) of the boson star, a necessary condition for scalarization to occur, and we derive an analytical expression for computing the sensitivity of a boson star in Brans-Dicke theory from its mass and particle number. Finally we comment on how one can use the sensitivity of a star in any Scalar-Tensor theory to determine how its mass changes when it undergoes gravitational evolution.

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1 Introduction

Scalar Tensor (ST) theories of gravity are the most natural generalisations of General Relativity (GR) and describe gravity as being mediated by both a metric field g_{ab} and a scalar field Φ . In the simplest ST theories, Newton's constant G is replaced by a field $G\Phi^{-1}$ at the level of the action, while Φ obeys a wave equation and is coupled to the curvature via a function $\omega(\Phi)$. The strength of this coupling decreases as ω increases and the special case of constant ω gives the Brans Dicke (BD) theory. One recovers GR by taking the limit $\omega \rightarrow \infty$, $\Phi \rightarrow \Phi_{GR}$, where Φ_{GR} is constant. By choosing appropriate units, one can set $\Phi_{GR} = 1$. In the original (Jordan frame) formulation of these theories, the ordinary matter is universally coupled to the curvature, so these theories are metric theories. As is well known, a conformal transformation may be used to express the theory in a new set of variables $g_{ab}^* = \Phi g_{ab}$, $\varphi = \varphi(\Phi)$, where g_{ab}^* is the Einstein frame metric and φ is a new scalar field that plays the role of a matter source and also couples the matter to g_{ab}^* . The Einstein frame representation is often more convenient mathematically than the Jordan frame (for example, the Cauchy problem is well posed in the Einstein frame [1]) although the Jordan frame is often considered to be the physical one.

At present, the primary motive for studying ST gravity comes from supergravity and superstring theories whose low energy limits always include one or more scalar fields (dilaton fields) that play a similar role to the Φ field of ST gravity. However, the coupling between the non-dilaton terms and the curvature in string actions is non-universal so that the correspondence

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between the low energy limit of supersymmetric theories and ST gravity is only approximate. Nevertheless, this approximation is fairly accurate since, to date, highly sensitive experiments have detected no violation of the principle of universal coupling [2].

In general, the results of Solar System experiments place the strongest constraints on the viability of most theories of gravity. For metric theories, these experiments are interpreted within the framework of the parameterised post-Newtonian (PPN) formalism which describes the gravitational field of the Solar System as an expansion about flat spacetime in powers of the Newtonian gravitational potential U near to the surface of the Sun. The coefficients appearing in the expansion up to the order U^4 terms, the so called first post Newtonian (1PN) order, are the 10 PPN parameters. In ST gravity only two of these are non-zero: γ and β , while Φ is everywhere close to some cosmologically determined boundary value Φ_B . In terms of the coupling function ω the two PPN parameters of ST theory are given by

$$\gamma = \frac{1 + \omega_B}{2 + \omega_B}, \quad \beta = 1 + (2\omega_B + 3)^{-2}(2\omega_B + 4)^{-1} \frac{d\omega}{d\Phi}, \quad (1)$$

where $\omega_B = \omega(\Phi_B)$ and $d\omega/d\Phi$ is to be evaluated at $\Phi = \Phi_B$. The observational constraints on γ and β are [2]

$$|\gamma - 1| \leq 0.0003, \quad |\beta - 1| \leq 0.002. \quad (2)$$

For BD theory, the first of these inequalities implies that $\omega \geq 3300$ and all of the predictions of BD theory differ from those of GR to within a relative error of $\sim 1/\omega$. For other ST theories, the observational data only places limits on the behaviour of $\omega(\Phi)$ in the slow motion, weak field limit.

A framework for analysing the motion of compact relativistic bodies in metric theories of gravity, the modified EIH formalism, has been developed (see Eardley [3] and Will [4] for details). The formalism treats each body as a point mass moving in the gravitational field of its companion bodies. Each body has an inertial mass which governs its response to the ambient gravitational field via quasi-Newtonian equations of motion. The role of non-metric gravitational fields (such as the Φ field in ST gravity) is to modify the inertial mass. The modified EIH formalism forms the basis of the post Keplerian formalism, which gives a parameterised description of the motion of binary pulsar systems and is used to interpret observational data gathered from these systems. These data must be used to constrain or rule out theories of gravity on a case by case basis (in contrast with solar system observations which can place limits on entire classes of theories by limiting the values of the PPN parameters).

Since the coupling function ω is completely arbitrary, there are an infinite number of viable ST theories. This leaves open the possibility that gravity is described by a ST theory whose predictions are arbitrarily close to those of GR at the current epoch and in the weak field limit, but differ considerably in strong field situations or at earlier cosmological epochs. Thus astrophysical objects with strong internal gravitational fields may still exhibit behaviour that differs greatly from that predicted by GR.

One strong field effect is spontaneous scalarization, recently discovered in neutron stars by Damour & Esposito-Farese [5, 6]. They found that, for ST theories in which the coupling function obeys the inequality

$$\tilde{\beta} := \frac{2\Phi_B}{(2\omega + 3)^2} \frac{d\omega}{d\Phi} \bigg|_{\Phi=\Phi_B} < -4 \quad (3)$$

and satisfies the constraints of eqn (2), the Einstein frame scalar field inside a neutron star rapidly becomes inhomogeneous once the star's mass increases above some critical value. For

a star whose mass is below this value, φ is nearly constant throughout the star (a state which minimises the star's energy), while for higher mass stars, the energy is minimised when φ has a large spatial variation. These effects become more pronounced in the limit $\Phi_B \rightarrow \Phi_{GR}$, $\omega_B \rightarrow \infty$. The coupling function they chose was of the form $2\omega + 3 \sim 1/(\log \Phi)$ which diverges as $\Phi \rightarrow \Phi_{GR} = 1$ and gives weak field neutron stars that are indistinguishable from their GR counterparts. A similar study was performed by Salgado, Sudarsky & Nucamendi [7], who considered neutron stars in a theory with a non-linearly coupled scalar field χ which is equivalent to standard ST theory with $\omega \sim \Phi/(\Phi - 1)$, which also diverges as $\Phi \rightarrow \Phi_{GR}$. Independently of Damour & Esposito-Farese, a study of strong field effects in ST neutron stars was carried out by Zaglauer [8]. However, the formalism developed in [8] involved an artificial definition of the sensitivity (defined below) of a neutron star, leading to the apparent occurrence of strong ST effects for choices of ω and Φ_B that the study in [5] showed should not occur.

It turns out that when spontaneous scalarization effects are present in neutron stars, the modified EIH formalism breaks down. An alternative, more sophisticated formalism, adapted exclusively to tensor-multi-scalar theories of gravity, has recently been devised by Damour & Esposito-Farese [1]. Their formalism, which is based on an expansion of an effective coupling strength α between the φ field and the non-scalar matter about its weak field value, is well suited to the study of the scalarization phenomenon.

The purpose of the present paper is twofold. First we discuss some aspects of the modified EIH formalism as it applies to ST gravity after introducing appropriate definitions of the inertial mass and sensitivity of a star. We then numerically compute the sensitivity of boson stars for several choices of ω . Secondly, we demonstrate the existence of spontaneous scalarization in boson stars and compare the behaviour of the sensitivity and the coupling parameter α with their behaviour in neutron stars.

2 Field Equations and Scalar Field Boundary Values

The Jordan frame action for scalar-tensor gravity universally coupled to one or more matter fields μ is given by

$$I = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi} \left(\Phi R - \frac{\omega(\Phi)}{\Phi} g^{ab} \partial_a \Phi \partial_b \Phi \right) + L_m(g^{ab}, \mu) \right] \quad (4)$$

where g_{ab} is the Jordan frame metric, R is the Ricci scalar curvature formed from g_{ab} and we use units in which $G = c = 1$. The generalised Einstein equations in this frame are found by varying I with respect to g^{ab} and are given by

$$G_{ab} = \frac{1}{\Phi} (\nabla_a \nabla_b \Phi - g_{ab} g^{cd} \nabla_c \nabla_d \Phi) + \frac{\omega}{\Phi^2} (\partial_a \Phi \partial_b \Phi - \frac{1}{2} g_{ab} g^{cd} \partial_c \Phi \partial_d \Phi) + \frac{8\pi}{\Phi} T_{ab} \quad (5)$$

where

$$T_{ab} := -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{ab}} \quad (6)$$

is the matter energy-momentum tensor, G_{ab} is the Einstein tensor formed from g_{ab} and ∇_a is the covariant derivative compatible with g_{ab} . The scalar and matter fields obey the equations

$$g^{ab} \nabla_a \nabla_b \Phi = \frac{1}{2\omega + 3} \left(8\pi T - \frac{d\omega}{d\Phi} g^{ab} \partial_a \Phi \partial_b \Phi \right), \quad \nabla_a T^{ab} = 0 \quad (7)$$

where $T := g^{ab} T_{ab}$.

To rewrite the field equations in the Einstein frame, we make the field redefinitions

$$g_{ab}^* = A^{-2} g_{ab}, \quad g_*^{ab} = A^2 g^{ab}, \quad d\varphi = \frac{\sqrt{2\omega+3}}{2\Phi} d\Phi, \quad (8)$$

where

$$A^2(\varphi) := \Phi^{-1} \quad (9)$$

is found by integrating the third of eqns (8) and solving for Φ . In the Einstein frame, the generalised Einstein equations are

$$G_{ab}^* = 2\partial_a\varphi\partial_b\varphi - g_{ab}^*g_*^{cd}\partial_c\varphi\partial_d\varphi + 8\pi T_{ab}^* \quad (10)$$

where

$$T_{ab}^* := A^2 T_{ab}, \quad (11)$$

while the matter and scalar field equations are

$$g_*^{ab}\nabla_a^*\nabla_b^*\varphi = -\frac{4\pi}{A}\frac{dA}{d\varphi}T^*, \quad \nabla_a^*T_*^{ab} = -\frac{16\pi}{A}\frac{dA}{d\varphi}\partial_a\varphi T_*^{ab}, \quad (12)$$

where $T^* := g^{ab}T_{ab}^*$ and ∇_a^* is the covariant derivative compatible with g_{ab}^* .

We analyse the behaviour of boson stars for three particular coupling functions ω . The first is the BD coupling, where ω is constant. The second is the exponential coupling law chosen in [5], which in the Einstein frame is

$$A(\varphi) = \exp(-k\varphi^2), \quad (13)$$

where k is a constant. This is equivalent to the Jordan frame coupling

$$2\omega + 3 = \frac{1}{2k \log \Phi}. \quad (14)$$

Our third choice is equivalent to that made by Salgado *et al*, who consider a theory in which the Lagrangian is given by

$$L = \frac{1}{16\pi} \left[(1 + \xi\chi^2)R - \frac{1}{2}g^{ab}\partial_a\chi\partial_b\chi \right] + L_m, \quad (15)$$

where ξ is a constant and χ a non-linearly coupled scalar field which is constrained to have the value $\chi \geq 0$. Making the field redefinition $1 + \xi\chi^2 = \Phi$, one finds that the theory derived from eqn (15) is the standard ST theory outlined above with the coupling function

$$\omega = \frac{\Phi}{8\xi(\Phi-1)}, \quad (16)$$

where $\Phi \geq 1$. It is more natural to choose Φ instead of χ as the physical scalar gravitational field since, when studying the motion of test bodies about a gravitating central source, it is Φ that appears explicitly in the equations of motion for the test bodies.

We consider asymptotically flat solutions in which Φ tends towards some limiting value Φ_∞ at spacelike infinity. We assume that a star embedded in a cosmological background can be approximated by an asymptotically flat solution and we identify Φ_∞ with Φ_B at the present epoch. Then equation (2) places restrictions on the maximum allowed value of Φ_∞ for any

solution at the current epoch. Using eqn (1), one finds that for the coupling given by eqn (14), the maximum allowed boundary value is given by the lower of the two values

$$\Phi_{\infty}^{max} = 1 + \frac{0.0003}{4k}, \quad \Phi_{\infty}^{max} = 1 + \frac{0.002}{2k^2}. \quad (17)$$

Similarly, for the coupling function given by eqn (16), the limiting value of Φ_{∞} is given by the lower of the two values

$$\Phi_{\infty}^{max} = 1 + \frac{0.0003}{8\xi}, \quad \Phi_{\infty}^{max} = 1 + \frac{0.002}{8\xi^2}. \quad (18)$$

3 Mass and Scalar Charge

To study the mass of a boson star, or any other stellar object, we may consider it either as a central source of gravitational field and explore the surrounding spacetime with test particles of differing compositions, or consider the star to be an extended, self-gravitating test body and examine its motion in the gravitational field of a second massive body. Either situation allows one to define several related quantities based upon the mass of the star and the behaviour of Φ in the far field region. The former situation was assumed by Salgado *et al.* [7], who examined the mass and scalar charge (defined below) of neutron stars considered as central sources. The latter situation is described by the modified EIH formalism and the formalism devised by Damour and Esposito-Farese and is considered by them in [5, 6]. We shall briefly summarise definitions of mass and scalar charge for both situations, as applied to any compact object (such as a boson star or neutron star), and outline part of the modified EIH formalism. We specialise to systems consisting of spherically symmetric bodies each of whose line elements, when described in the Jordan frame, are given by

$$ds^2 = -B(r)dt^2 + \frac{1}{1 - 2m(r)/r}dr^2 + r^2dS^2 \quad (19)$$

where dS^2 is the line element of the unit 2-sphere and $m(r)$ is the usual generalised Schwarzschild mass. Each body is modelled as a static, asymptotically flat object.

Consider first the motion of test particles about a central star. We define the Jordan frame scalar charge of the star by

$$Q_S := \lim_{r \rightarrow \infty} \left[r^2 \frac{d\Phi}{dr} \right]. \quad (20)$$

Taking the ADM mass $M_{ADM} := \lim_{r \rightarrow \infty} m(r)$ together with the scalar charge, we define the tensor mass

$$M_T := M_{ADM} - \frac{Q_S}{2\Phi_{\infty}} \quad (21)$$

where $\Phi_{\infty} := \lim_{r \rightarrow \infty} \Phi$ is the asymptotic value of the gravitational scalar field. M_T measures the total energy of the star and is also the active gravitational mass measured by an orbiting test particle for which Φ is locally homogeneous (such as a small black hole). An orbiting test particle with negligible self-energy moves on geodesics of g_{ab} and measures the Keplerian mass

$$M_K := M_{ADM} - \frac{Q_S}{2\Phi_{\infty}}. \quad (22)$$

This is different from M_T since the Strong Equivalence Principle (SEP, defined in [4]) is violated in ST gravity.

From the expressions for M_K and M_T we have

$$Q_S = 2\Phi_\infty(M_T - M_K). \quad (23)$$

A locally freely falling Cavendish experiment far from the star measures a gravitational coupling strength

$$\tilde{G} = \Phi_\infty^{-1} \frac{2\omega_\infty + 4}{2\omega_\infty + 3} \quad (24)$$

where $\omega_\infty = \omega(\Phi_\infty)$. Given a particular form of ω , one can then calculate Φ_∞ . Hence by combining the above two equations, we find that Q_S is measurable in principle.

Salgado *et al.* define and use the scalar charge

$$\tilde{Q}_S = \lim_{r \rightarrow \infty} \left[r^2 \frac{d\chi}{dr} \right]. \quad (25)$$

associated with the field χ that appears in the Lagrangian (15). This quantity is related to the charge Q_S associated with Φ by

$$Q_S = 2\xi\chi\tilde{Q}_S = 2\tilde{Q}_S\sqrt{\xi(\Phi_\infty - 1)}. \quad (26)$$

From the above definitions it appears more natural to use Q_S as the physical scalar charge of the star instead of \tilde{Q}_S .

Both M_T and Q_S may be related to corresponding Einstein frame quantities as follows. In the Einstein frame, the Schwarzschild radial coordinate r^* is given by $r^* = \sqrt{\Phi}r$. Then, defining the Einstein frame scalar charge Q_S^* by

$$Q_S^* := \lim_{r^* \rightarrow \infty} \left(r^{*2} \frac{d\varphi}{dr^*} \right) \quad (27)$$

and using the third of eqns (8) gives

$$Q_S^* = \frac{\sqrt{2\omega_\infty + 3}}{2\sqrt{\Phi_\infty}} Q_S. \quad (28)$$

One can also show that M_T is related to the Einstein frame ADM mass M_{ADM}^* by

$$M_{ADM}^* = \sqrt{\Phi} M_T. \quad (29)$$

Consider now the motion of a spherical star, with some conserved particle number N , in the gravitational field of a companion body. We assume that the separation between the two bodies is much greater than either of their radii, and we denote the inter body value of the scalar field by Φ_∞ . The scalar field Φ_L in the interior of the star must be matched smoothly to Φ_∞ and its value will affect the structure of the star. In particular, its inertial mass M_I and hence its motion in the gravitational field of the companion body will be sensitive to changes in the value of Φ_∞ . Expanding Φ_L about Φ_∞ as $\Phi_L = \Phi_\infty + \delta\Phi$, we write

$$M_I = M_{I_\infty} + \frac{\partial M_I}{\partial \Phi} \delta\Phi + \mathcal{O}(\delta\Phi^2), \quad (30)$$

where M_{I_∞} is the value of M_I when $\Phi_L = \Phi_\infty$, or

$$M_I = M_{I_\infty} \left[1 + s \frac{\delta\Phi}{\Phi_\infty} \right] + \mathcal{O}(\delta\Phi^2), \quad (31)$$

where s is the first sensitivity defined by

$$s := \frac{\partial(\log M_I)}{\partial(\log \Phi)}. \quad (32)$$

In eqns (30) and (32), the derivatives are evaluated at $\Phi = \Phi_\infty$ for fixed N . Higher order sensitivities may be defined by taking successively higher order derivatives of M_I , to obtain the expansion for M_I to higher powers of $\delta\Phi$. s measures the response of a star's inertial mass to changes in the value of Φ_∞ and it also gives a useful measure of the compactness of the star [1].

Here we define M_I to be

$$M_I := \Phi_\infty M_T. \quad (33)$$

This differs from the definition given by Will [4] by a factor of $(2\omega_\infty + 3)/(2\omega_\infty + 4)$ and is chosen so that our definition corresponds with the Einstein frame quantity defined by Damour & Esposito-Farese in [1]. The difference in definitions of M_I does not affect the BD analysis given in [4] since, for constant ω , this factor cancels in the definition of s given there. For more general ST theories, our definition of M_I leads to a self consistent set of equations of motion that, to 1PK order, are identical to the BD equations derived in [4] (except for the different definition of M_I) and obviates the need to introduce the modification of s given in [8].

These definitions form the basis of the modified EIH formalism, examined in detail for more general theories of gravity in [4], and s and its derivatives play a fundamental role in determining the motion of a body in the gravitational field of other bodies. Formally, to derive the equations of motion governing such a system, one first determines the inter body gravitational fields from the field equations where the M_I appear as delta function terms in T_{ab} . One then substitutes the solution into a quasi-Newtonian particle Lagrangian, taking into account changes in the M_I due to the relative velocities of the bodies, to find the equations of motion for each body. These equations explicitly involve the sensitivities of the bodies, which determine how the motion deviates from the Newtonian prediction, and are written as an expansion in powers of v/c about the Newtonian solution, where v is a characteristic velocity of the system. (One cannot expand in powers of the Newtonian potential of the bodies since for strongly bound objects, U is of the order of 1). The expansion is normally taken to order v^4/c^4 , the first post Keplerian (1PK) order, and in the limit that U for each body becomes small, the equations of motion reduce to the 1PN equations. The post Keplerian equations then are a parameterised description of the modified EIH equations of motion, adapted so that the system is described directly in terms of observable parameters.

For a system composed of two bodies it turns out that only the first sensitivity s appears in the equations of motion to 1PK order. For example, as shown in detail in [4], given a star of inertial mass M_I^A and sensitivity s_A with a companion of mass M_I^B and sensitivity s_B at a distance R_{AB} , the quasi-Newtonian gravitational force F between the two bodies has a magnitude $F = G_{AB} M_I^A M_I^B / R_{AB}^2$ where the effective gravitational coupling strength G_{AB} is

$$G_{AB} = \frac{2\omega_\infty + 4}{\Phi_\infty(2\omega_\infty + 3)} \left[1 - \frac{1}{\omega_\infty + 2} (s_A + s_B - 2s_A s_B) \right] \quad (34)$$

to 1PK order. (Note that the above equation is a simplification that ignores effects due to the relative velocity of the two bodies.) If body B is a non-self-gravitating test particle then $s_B = 0$ and body A has a Kepler measured mass

$$M_K^A = \frac{F R_{AB}}{M_I^B} = \frac{(2\omega_\infty + 4) M_I^A}{\Phi_\infty(2\omega_\infty + 3)} \left(1 - \frac{s_A}{\omega_\infty + 2} \right). \quad (35)$$

In ST gravity, the Φ field outside an asymptotically flat static black hole is constant and one can show that the sensitivity s_{BH} of a black hole is $s_{BH} = \frac{1}{2}$. Assuming that a small black hole in orbit about body A also has this property, then the orbital mass in this case is given by

$$\frac{(2\omega_\infty + 4)M_I^A}{\Phi_\infty(2\omega_\infty + 3)} \left(1 - \frac{1}{2\omega_\infty + 4}\right) = \frac{M_I^A}{\Phi_\infty} \quad (36)$$

which consistent with our relation (33) between the inertial and tensor masses.

The formalism devised by Damour & Esposito-Farese [1] for tensor-multi-scalar theories is similar to the modified EIH formalism except that it describes the motion of the system in terms of Einstein frame variables. In place of s , the fundamental quantity used is a coupling parameter α which, for a star of mass M_{ADM}^* and Einstein frame scalar charge Q_S^* , is defined by

$$\alpha := \frac{Q_S^*}{M_{ADM}^*} \quad (37)$$

and measures the strength of the coupling between φ and the non-scalar matter. In the weak field limit, $\alpha \rightarrow -1/\sqrt{2\omega_\infty + 3}$. The equations of motion at the 1PK level the depend upon the values of α and $d\alpha/d\varphi$, evaluated at $\varphi = \varphi_\infty$.

Damour & Esposito-Farese (see Appendix A of [1]) have shown that, by varying the Einstein frame Hamiltonian with respect to changes in the asymptotic value of the scalar field φ , one can derive the following relation:

$$\frac{Q_S^*}{M_{ADM}^*} = \frac{1}{M_{ADM}^*} \frac{\partial M_{ADM}^*}{\partial \varphi} \quad (38)$$

where the derivative is evaluated at $\varphi = \varphi_\infty$ for fixed particle number. A similar analysis, performed on the Jordan frame Hamiltonian, gives

$$\frac{Q_S}{M_I} = \frac{-2}{2\omega_\infty + 3}(1 - 2s). \quad (39)$$

Using eqn (39) one can calculate s directly from the scalar charge and inertial mass. Combining eqns (38) and (39) and using eqns (29) and (33), we find that the parameters α and s are related by

$$\alpha = \frac{-1}{\sqrt{2\omega_\infty + 3}}(1 - 2s), \quad (40)$$

to 1PK order.

Formally ST gravity reduces to GR in the limit $\omega \rightarrow \infty$, $\Phi \rightarrow \Phi_{GR} = 1$ (in units with $G = 1$ and provided $g^{ab}T_{ab}$ is non-zero). In this limit, s generally approaches a non-zero value and measures the response of M_I to a (global) change in the value of Newton's constant G .

The modified EIH formalism is adequate for describing the motion of compact bodies when effects such as spontaneous scalarization are absent. However, as shown in [5, 6], when scalarization occurs in a neutron star, both α and Q_S^* remain finite and non-zero as $\omega_\infty \rightarrow \infty$. Equation (40) then implies that s must diverge and so the modified EIH formalism breaks down. We shall discuss this further in Section 5, where we also show how s behaves for various boson star models.

4 Boson Stars

Boson stars are gravitationally bound configurations of zero temperature bosons and many of their properties in ST gravity have already been studied [9, 10, 11, 12, 13]. They share many

features in common with simple neutron star and white dwarf solutions. Sequences of boson star solutions may be parameterised by the central boson field density, and when plotted against this parameter, the mass and conserved charge curves show a maximum at which instability first occurs.

We take as a matter source a complex self-interacting boson field Ψ with Lagrangian

$$L_m = -\frac{1}{2}g^{ab}(\partial_a\Psi\partial_b\bar{\Psi} + \partial_a\bar{\Psi}\partial_b\Psi) - \Psi\bar{\Psi} - 4\pi\Lambda(\bar{\Psi}\Psi)^2 \quad (41)$$

where Λ measures the strength of the boson self-interaction and we have chosen units in which $\hbar/\mu = 1$, where μ is the boson mass. The Jordan frame energy momentum tensor for this field is

$$T_{ab} = \partial_a\Psi\partial_b\bar{\Psi} + \partial_a\bar{\Psi}\partial_b\Psi - \frac{1}{2}g_{ab}g^{cd}(\partial_c\Psi\partial_d\bar{\Psi} + \partial_c\bar{\Psi}\partial_d\Psi) - g_{ab}\Psi\bar{\Psi} - 4\pi g_{ab}\Lambda(\bar{\Psi}\Psi)^2 \quad (42)$$

We consider static, spherical solutions with the line element (19) and we only consider minimum energy solutions, which implies that the boson field wave function may be written as

$$\Psi = \frac{P(r)}{\sqrt{8\pi}} \exp(i\Omega t) \quad (43)$$

(see Appendix) where $P(r)$ is a real dimensionless function and the constant Ω is real. For the metric given implicitly in the line element (19), the independent components of the field and matter equations (5) and (7) are

$$m' = \frac{r}{2\omega + 3} \left[(r - 2m) \left(\frac{\Phi'^2}{2\Phi} \frac{d\omega}{d\Phi} + \frac{P'^2}{2\Phi} (1 + 2\omega) \right) + \frac{rP^2}{2\Phi} \left(2\omega - 1 + (2\omega + 5) \frac{\Omega^2}{B} \right. \right. \\ \left. \left. + (2\omega - 1) \frac{P^2\Lambda}{2} \right) \right] + (r - 2m) \left[\frac{r\Phi'B'}{4B\Phi} + \frac{\Phi'^2 r\omega}{4\Phi^2} \right], \quad (44)$$

$$B' = \frac{1}{2\Phi + r\Phi'} \left[\frac{\Phi'^2 \omega B r}{\Phi} - 4B\Phi' + 2P'^2 B r - \frac{1}{r - 2m} \left(2P^2 r^2 (B - \Omega^2) + P^4 B \Lambda + \frac{4Bm\Phi}{r} \right) \right], \quad (45)$$

$$\Phi'' = \Phi' \left[\frac{rm' - m}{r(r - 2m)} - \frac{2}{r} - \frac{B'}{2B} \right] - \frac{2}{2\omega + 3} \left[\frac{\Phi'^2}{2} \frac{d\omega}{d\Phi} + P'^2 + \frac{P^2 r}{r - 2m} \left(2 - \frac{\Omega^2}{B} + 2\Lambda P^2 \right) \right] \quad (46)$$

and

$$P'' = P' \left[\frac{rm' - m}{r(r - 2m)} - \frac{2}{r} - \frac{B'}{2B} \right] + \frac{Pr}{r - 2m} \left[1 - \frac{\Omega^2}{B} + \Lambda P^2 \right], \quad (47)$$

where a prime denotes d/dr . For the corresponding Einstein frame equations, see [11].

The existence of a global $U(1)$ symmetry in the matter Lagrangian (41) implies the existence of a conserved charge N . Using the above coordinates and field variables, this may be written as

$$N = \int_0^\infty \frac{r^2 \Omega P^2}{\sqrt{B} \sqrt{1 - 2m/r}} dr \quad (48)$$

and is interpreted as the total number of bosons in the star.

Equations (44) to (47) must be integrated numerically. To ensure that the solutions describe bound eigenstates of Ψ one must impose the boundary condition

$$\lim_{r \rightarrow \infty} P(r) = 0 \quad (49)$$

and the regularity conditions

$$m_0 = 0, \quad P'_0 = 0, \quad \Phi'_0 = 0, \quad (50)$$

where the subscript 'o' denotes values at the origin. The field equations then become eigenvalue equations for Ω , are parameterised by P_0 for fixed Φ_∞ and automatically lead to the boundary conditions $\Phi = \Phi_\infty + \mathcal{O}(r^{-1})$, $m(r) = M_{ADM} + \mathcal{O}(r^{-1})$ and $B = B_\infty + \mathcal{O}(r^{-1})$ as $r \rightarrow \infty$. For any solution, one can make the rescaling $B_\infty \rightarrow 1$ by rescaling Ω appropriately.

In Brans-Dicke theory, where $d\omega/d\Phi = 0$, the equations with $\Lambda = 0$ are invariant under the rescaling

$$P \rightarrow \kappa P, \quad \Phi \rightarrow \kappa^2 \Phi \quad (51)$$

where κ is a constant. Equation (51) leaves M_T , M_K and M_{ADM} invariant and rescales the particle number as

$$N \rightarrow \kappa^2 N. \quad (52)$$

We can use this scaling invariance to find an alternative expression of s in a $\Lambda = 0$ BD boson star as follows. Consider a pair of solutions, σ_1 and σ_2 , with the same boundary value Φ_∞ and with central boson field amplitudes $P_0^{(1)}$ and $P_0^{(2)} = P_0^{(1)} + \delta P_0$, such that $\delta P_0 \ll 1$. Solution σ_1 has mass $M_T^{(1)}$ and particle number $N^{(1)}$, while σ_2 has mass $M_T^{(2)}$ and particle number $N^{(2)}$. Then, to first order in δP_0 ,

$$M_T^{(2)} = M_T^{(1)} + \frac{\partial M_T}{\partial P_0} \delta P_0 \quad (53)$$

and

$$N^{(2)} = N^{(1)} + \frac{\partial N}{\partial P_0} \delta P_0, \quad (54)$$

where the derivatives are taken with Φ_∞ held fixed. We use eqn (51) to generate a new solution $\tilde{\sigma}_2$ with mass $\tilde{M}_T^{(2)}$, particle number $\tilde{N}^{(2)}$ and boundary scalar field $\tilde{\Phi}_\infty$ and we choose a value of κ such that $\tilde{N}^{(2)} := \kappa^2 N^{(2)} = N^{(1)}$. From eqn (54), κ^2 must then be given by

$$\kappa^2 = 1 - \frac{1}{N^{(1)}} \frac{\partial N}{\partial P_0} \delta P_0 \quad (55)$$

to first order in δP_0 . This implies that

$$\tilde{\Phi}_\infty = \kappa^2 \Phi_\infty = \Phi_\infty \left(1 - \frac{1}{N^{(1)}} \frac{\partial N}{\partial P_0} \delta P_0 \right), \quad (56)$$

while $M_T^{(2)}$ is invariant under this rescaling, thus $\tilde{M}_T^{(2)} = M_T^{(2)}$. Then, using eqn (53), we have

$$\delta M_T := \tilde{M}_T^{(2)} - M_T^{(1)} = -\frac{\partial M_T}{\partial P_0} \delta P_0 \quad (57)$$

and, from eqn (56),

$$\delta \Phi_\infty := \tilde{\Phi}_\infty - \Phi_\infty = -\frac{\Phi_\infty}{N^{(1)}} \frac{\partial N}{\partial P_0} \delta P_0. \quad (58)$$

Combining these results with the definition (33) gives

$$\frac{\delta M_I}{\delta \Phi_\infty} = \frac{\delta(\Phi_\infty M_T)}{\delta \Phi_\infty} = M_T^{(1)} - \Phi_\infty \frac{\partial M_T}{\partial P_0} \delta P_0 \left[\frac{\Phi_\infty}{N^{(1)}} \frac{\partial N}{\partial P_0} \delta P_0 \right]^{-1} \quad (59)$$

$$= M_T^{(1)} - N^{(1)} \frac{\partial M_T}{\partial N}. \quad (60)$$

Taking the limit $\delta\Phi_\infty \rightarrow 0$ and using eqn (32) we have

$$s = \frac{\Phi_\infty}{M_I} \left(M_T - N \frac{\partial M_T}{\partial N} \right) = 1 - \frac{N}{M_T} \frac{\partial M_T}{\partial N}, \quad (61)$$

where we have dropped the label from N and M_T since this relation is true for any initial solution σ_1 . One can show [12] that for fixed Φ_∞

$$\frac{\partial M_T}{\partial N} = \frac{\Omega}{\Phi_\infty}. \quad (62)$$

Combining this result with eqn (61) gives an alternative expression for s :

$$s = 1 - \frac{\Omega N}{\Phi_\infty M_T}, \quad (63)$$

Note that eqns (61) and (63) hold for any value of ω .

When Λ is non-zero but finite, one cannot drive a similar result since one also needs to rescale Λ to keep the field equations invariant. However, Gunderson & Jensen [9] have shown that in BD theory, even a relatively small value of Λ causes the terms that are quartic in P to dominate the energy momentum tensor. For $\Lambda > 100$ the solutions are well approximated by taking the limit $\Lambda \rightarrow \infty$, and one can derive an expression for the sensitivity of a boson star in this limit. One first rewrites the field equations by making the field and coordinate re-definitions $\Pi = \sqrt{\Lambda}P$, $\rho = r/\sqrt{\Lambda}$ and then takes the limit $\Lambda \rightarrow \infty$. The resulting equations are identical to eqns (44) to (47) except that P and r are replaced by Π and ρ , $\Lambda = 1$, all terms including a factor of Π' vanish and Π is given by the algebraic expression

$$\Pi^2 = \frac{\Omega^2}{B} - 1. \quad (64)$$

See [9] for details in BD theory. The equations defining N , Q_S and M_T are identical to the finite Λ equations, except that P and r are everywhere replaced by Π and ρ . Then masses are measured in units of M_{pl}^3/μ^2 and in BD theory the new field equations are invariant under the rescaling

$$\Phi \rightarrow \kappa^2 \Phi, \quad \rho \rightarrow \kappa \rho, \quad (65)$$

where κ is constant. Under eqn (65), the mass and particle number rescale as

$$M_T \rightarrow \kappa M_T, \quad N \rightarrow \kappa^3 N. \quad (66)$$

Performing a similar analysis to the one given above for the $\Lambda = 0$ case and using eqns (65) and (66), one can show that the sensitivity in the limit $\Lambda \rightarrow \infty$ is given by

$$s = \frac{3}{2} \left(1 - \frac{N}{M_T} \frac{\partial M_T}{\partial N} \right) = \frac{3}{2} \left(1 - \frac{N\Omega}{\Phi_\infty M_T} \right). \quad (67)$$

The first of these equalities is identical to the result derived for neutron stars [3] (in the latter case, N must be replaced by the neutron star's conserved baryon number). This is because both neutron stars and boson stars in the limit $\Lambda \rightarrow \infty$ have masses that scale as M_{pl}^3/μ^2 , where for the neutron star μ is the baryon mass. in contrast, the mass of a boson star with no self-interaction scales as M_{pl}^2/μ , so there is no factor of 3/2 in eqns (61) and (63).

One cannot carry out the same analysis for more general ST theories since the ST versions of eqns (51) and (65) require $\omega(\Phi)$ to be held fixed. Since Φ rescales under eqn (51), the functional form of ω must change to compensate in a way that depends upon the precise choice of ω . However, eqns (61) and (67) will still be approximately correct for solutions in which $d\omega/d\Phi$ remains small.

5 Equilibrium Solutions

We have numerically integrated the field equations (44) to (47) for BD boson stars and for ST boson stars with the coupling functions given by eqns (14) and (16). A feature shared by all solutions sets it that both M_T and N increase from zero at $P_0 = 0$ to reach coinciding maxima $M_T^{(max)}$ and $N^{(max)}$ at some value of $P_0 = P_0^{(max)}$. For $P_0 > P_0^{(max)}$, the solutions become unstable. The magnitude of the scalar charge $|Q_S|$ also increases from zero at $P_0 = 0$ but its first maximum $|Q_S|^{(max)}$ does not occur at the same value of P_0 as the maxima in N and M_T . For all solutions discussed here, we have found that $s < 0.5$ which implies that Q_S , α and Q_S^* are all negative.

We consider first boson stars in BD theory. One can show that in the weak field limit, where the (dimensionless) parameter $P_0 \rightarrow 0$, the sensitivity is given by

$$s = -2\mathcal{E} + \mathcal{O}(P_0^2), \quad (68)$$

where \mathcal{E} is the fractional binding energy defined by as

$$\mathcal{E} := \frac{\Phi_\infty M_T - N}{N}. \quad (69)$$

For stable stars, $\mathcal{E} < 0$ and hence all stable BD stars have $s > 0$ in the weak field limit. Numerical calculations then show that s increases with P_0 to reach some maximum at a value of $P_0 > P_0^{(max)}$. Values of $s_{(max)}$, the sensitivity of the maximum mass stable solution at $P_0 = P_0^{(max)}$, along with corresponding values $\alpha_{(max)}$ of the coupling parameter α , are shown in Table 1 for several choices of ω for both $\Lambda = 0$ and in the limit $\Lambda \rightarrow \infty$.

$\Lambda \rightarrow \infty$			$\Lambda = 0$		
ω	$s_{(max)}$	$\alpha_{(max)}$	ω	$s_{(max)}$	$\alpha_{(max)}$
1	0.211	-0.258	1	0.144	-0.318
10	0.189	-0.130	10	0.126	-0.156
500	0.182	-0.020	500	0.121	-0.024
3300	0.178	-0.008	3300	0.119	-0.009

Table 1: Sensitivity s and coupling parameter α of maximum mass boson stars in BD gravity for $\Lambda = 0$ and in the limit $\Lambda \rightarrow \infty$.

As eqns (63) and (67) suggest, for each ω the sensitivity of a maximal mass boson star with $\Lambda \rightarrow \infty$ is greater than the sensitivity of the corresponding $\Lambda = 0$ star by a factor of 3/2. This is also true for values of N less than the maximum value. For both choices of Λ , $s_{(max)}$ decreases with increasing ω towards some non-zero limit as $\omega \rightarrow \infty$ and to the level of accuracy quoted in the Table, $s_{(max)}$ for a boson star in GR has the same value as for a $\omega = 3300$ star. For comparison, maximum mass neutron stars have been found to have sensitivities that vary from $s_{(max)} = 0.2$ to $s_{(max)} = 0.39$, depending upon the equation of state chosen [3, 6]. Hence for boson stars with no self-interaction, the response of M_I to changes in Φ is much smaller than for neutron stars, while for boson stars with $\Lambda \rightarrow \infty$, the sensitivity is almost comparable to that of a neutron star. Note that, from the relation (40), we have $\alpha \rightarrow 0$ as $\omega \rightarrow \infty$ since s remains finite in this limit for all BD solutions.

We next consider boson stars in more general ST theories. One can show that, for any ST theory and for any value of Λ , in the weak field limit s is given by

$$s = -\mathcal{E}(2 + (2\omega_\infty + 3)\tilde{\beta}) + \mathcal{O}(P_0^2). \quad (70)$$

From eqns (1) and the limits (2), we have for any weak field solution $s < 0$ when $\tilde{\beta} < -0.0003$.

Figure 1 shows sensitivity s against particle number N for several sets of boson star solutions with $\Lambda = 0$, for the coupling function (14) with the parameter choices $k = 1$ and $k = 3$. For this coupling function, $\tilde{\beta} = -2k$ and the inequality (3) is satisfied when $k > 2$. For the values of k we are using, it turns out that the first inequality of eqns (17) places the tightest constraints on the value of Φ_∞ and these are $(\Phi_\infty - 1) \leq 7.5 \times 10^{-5}$ for $k = 1$ and $(\Phi_\infty - 1) \leq 8.3 \times 10^{-6}$ for $k = 3$. This latter value corresponds to the limit $\varphi_\infty \leq 0.0012$ which is smaller than the limit $\varphi_\infty \leq 0.0043$ quoted in [5] since the constraint on the PPN parameter γ has recently been tightened [2]. For the sake of comparison, in the Figure we have included $k = 3$ solutions with the same boundary value used in [5], although we also choose values compatible with the current observational constraints.

The curves in Figure 2 show the coupling parameter α against N for the $k = 3$ solutions shown in Figure 1. In contrast with the corresponding neutron stars curves shown in [5], the maximum value of $-\alpha$ occurs when $P_0 > P_0^{(max)}$ or, equivalently, when $N < N^{(max)}$ and the solutions are unstable. This is because, for any set of boson star solutions, $|Q_S|^{(max)}$ occurs at a parameter value $P_0 > P_0^{(max)}$.

k	$\tilde{\beta}$	$\Phi_\infty - 1$	φ_∞	ω_∞	$s_{(max)}$	$\alpha_{(max)}$	$-Q_S^{(max)}$	$-Q_S^{*(max)}$
1	-2	7.5×10^{-5}	4.3×10^{-3}	6759	-0.0035	-0.00920	1.0×10^{-4}	0.0058
3	-6	1.1×10^{-4}	4.3×10^{-3}	750	-2.70	-0.165	5.3×10^{-3}	0.10
3	-6	1.1×10^{-6}	4.3×10^{-4}	75114	-4.00	-0.0232	7.6×10^{-5}	0.015
3	-6	1.1×10^{-8}	4.3×10^{-5}	7.51×10^6	-4.02	-0.00233	7.6×10^{-7}	0.0015
3	-6	1.1×10^{-10}	4.3×10^{-6}	7.51×10^8	-4.02	-0.000233	7.6×10^{-9}	0.00015

Table 2: Numerical data for boson stars with $\Lambda = 0$ and $2\omega + 3 = 1/(2k \log \Phi)$. The label (max) denotes values at the maximum mass solution.

Some of the data taken from the numerical calculations are shown in Table 2. Together with Figures 1 and 2, they show that the scalarization phenomenon does not occur in $\Lambda = 0$ boson stars. As the study in [5] shows, in the limit $(\Phi_\infty - 1) \rightarrow 0$ and for neutron stars with $\tilde{\beta} < -4$, the values of both $-\alpha$ and Q_S^* for a star of given baryon number approach some finite limit (which is either zero for stars whose mass is below some critical value or non-zero for stars whose mass is above this value). From eqn (40), this implies that, for stars above the critical mass, s is negative and also that s diverges as $\sqrt{\omega_\infty}$ in this limit. In addition, from eqn (28), Q_S vanishes as $1/\sqrt{\omega_\infty}$. In contrast, for boson stars without self-interaction we find that for all solutions s reaches some finite, non-zero limit as $(\Phi_\infty - 1) \rightarrow 0$. This implies that α must vanish in this limit for all solutions, as shown by eqn (40). As shown in Table 2, the magnitude of both $Q_S^{(max)}$ and $Q_S^{*(max)}$ decrease to zero as $\Phi_\infty - 1 \rightarrow 0$ and this is true in these solutions for all other values of N .

We next consider $\Lambda = 0$ boson stars with the coupling function (16). Assuming that $\Phi_\infty - 1$ is small, we have $\tilde{\beta} = -4\xi$ which implies that the inequality (3) is satisfied when $\xi > 1$. Figures 3 and 4 show curves of s and α against N for solutions with $\xi = 1$ and $\xi = 2$. For these choices of ξ , the first of eqns (18) places the tightest limits on the values of Φ_∞ , which are $(\Phi_\infty - 1) < 3.75 \times 10^{-5}$ for $\xi = 1$ and $(\Phi_\infty - 1) < 1.87 \times 10^{-5}$ for $\xi = 2$. In the Figures we have chosen boundary values compatible with these limits. The behaviour of the solutions is qualitatively similar to the behaviour of the solutions shown in Figures 1 and 2 and, again, the maximum value of $-\alpha$ occurs after N has reached its first maximum. Data taken from these calculations is shown in Table 3.

ξ	$\tilde{\beta}$	$\Phi_\infty - 1$	ω_∞	$s_{(max)}$	$\alpha_{(max)}$	$-Q_S^{(max)}$	$-Q_S^{*(max)}$
1	-4	3.75×10^{-5}	3329	-0.433	-0.0229	0.00035	0.014
1	-4	1.87×10^{-5}	6679	-0.438	-0.0162	0.00018	0.010
1	-4	1.91×10^{-6}	65536	-0.422	-0.0051	0.000018	0.0032
2	-8	1.87×10^{-5}	3339	-7.39	-0.193	0.0028	0.113
2	-8	9.41×10^{-6}	6637	-10.1	-0.184	0.0019	0.109
2	-8	2.38×10^{-7}	262144	-45.7	-0.128	0.00021	0.077

Table 3: Numerical data for boson stars with $\Lambda = 0$ and $\omega = \Phi/(8\xi(\Phi - 1))$.

Finally, we consider solutions to the field equations in the limit $\Lambda \rightarrow \infty$ for the coupling function (14) and for $k = 3$. As mentioned above, solutions with this limiting value of Λ are a good approximation of solutions having large but finite values of Λ in BD theory, and we expect this to be true for other ST theories. Figure 5 shows curves of α against N for these solutions and data taken from these integrations is shown in Table 4. In contrast with the $\Lambda = 0$ solutions, these solutions show the same scalarization phenomena as the neutron stars studied in [5, 7]. When the boundary value $(\Phi_\infty - 1) > 0$, the coupling parameter α is small and negative for small N and decreases smoothly as N increases to its first maximum. For these solutions, ω_∞ is finite and all have finite, but negative, sensitivities. In the limit $(\Phi_\infty - 1) \rightarrow 0$, $\omega_\infty \rightarrow \infty$, the solutions divide into two classes: for those with N below a critical value $N_c = 0.200N < N_c$, α vanishes while s decreases smoothly with increasing N to diverge at $N = N_c$. For stars with $N > N_c$, α is non-zero and increases rapidly with N , while s remains divergent. The star with particle number N_c mass $M_T = 0.189$. Note that Figure 5 shows that the transition point between small and large values of $-\alpha$ becomes sharper as $(\Phi_\infty - 1)$ decreases, just as is the case for neutron stars.

$\Phi_\infty - 1$	ω_∞	$s_{(max)}$	$\alpha_{(max)}$	$-Q_S^{(max)}$	$-Q_S^{*(max)}$
1.1×10^{-4}	750	-6.96	-0.385	3.51×10^{-3}	0.068
1.1×10^{-8}	7.51×10^6	-141	-0.073	7.09×10^{-6}	0.014
1.1×10^{-10}	7.51×10^8	-1125	-0.058	5.66×10^{-7}	0.011
1.1×10^{-12}	7.51×10^{10}	-10829	-0.056	5.45×10^{-8}	0.011

Table 4: Numerical data for boson stars in the limit $\Lambda \rightarrow \infty$ with $2\omega + 3 = 1/(2k \log \Phi)$ and for $k = 3$.

6 Discussion

We have analysed spontaneous scalarization in ST boson stars for several choices of the coupling function ω . We have found that scalarization does not occur when the bosons have no self-interaction since, in this case, the sensitivity (or compactness) of the stars is small. With the inclusion of a large quartic self-interaction, the stars become much more compact and spontaneous scalarization occurs. The fact that this phenomenon occurs for boson stars as well as for simple neutron star models suggests that scalarization may be a universal characteristic of ST gravity.

We have also given a brief introduction to the modified EIH formalism, in which the sensitivity plays an important role, and shown how one can calculate the sensitivity of a boson star (or any other compact object) in ST theory from its mass and scalar charge. As well as

giving an indication of the compactness of a star, the sensitivity also measures the response of the star's inertial mass to changes in the asymptotic value of Φ . Hence some of our results may also be applied to the study of gravitational evolution, in which the structure, and in particular the mass, of a star embedded in a cosmological background is forced to evolve as the value of the cosmological scalar field changes with cosmological time. This phenomenon was first studied for ST boson stars by Torres, Schunck & Liddle [14] who assumed that the star evolved quasi-statically and could be modelled as a sequence of asymptotically flat equilibrium solutions whose boundary value Φ_∞ matches the cosmological value of Φ (which in general is an increasing function of cosmological time). These results were extended in [13]. Gravitational evolution has also been shown to affect the cooling rate of white dwarfs, which provides a new method of constraining ST theories of gravity [15].

Given a value of s for a star one immediately knows how its inertial mass will evolve, given the assumptions made in [14]. For example, in BD theory we have found that s is positive for all boson star solutions, and is positive for all BD white dwarf and neutron star solutions that we are aware of. Hence M_I will be an increasing function of cosmological time for all of these objects. For other ST theories s may be negative so that M_I will be a decreasing function of time.

The tensor mass (as we have defined it here) will in general decrease with time for all ST theories. From the definitions (32) and (33) we have

$$\frac{\partial M_T}{\partial \Phi} = \frac{M_I}{\Phi_\infty}(s - 1) \quad (71)$$

where the derivative is evaluated at $\Phi = \Phi_\infty$. For a black hole $s = 0.5$ and, since a black hole is the most compact of all stellar objects, one would expect other bodies to have $s < 0.5$. This is certainly the case for the boson star solutions discussed here. Then eqn (71) implies that M_T will always be a decreasing function of time in any cosmological model in which Φ increases with cosmological time.

A Appendix: Minimum Energy Solutions

A proof that eqn (43) leads to minimum energy solutions for static, spherically symmetric boson stars can be found in [16]. However, as noted by Jetzer [17], no corresponding proof has been given for the case of a boson field with $U(1)$ charge. For completeness, we generalise the proof in [16] to include charge and to allow for arbitrary static, asymptotically flat spacetimes.

We start with the matter Lagrangian L_m for bosons carrying charge e , which reads

$$L_m = -\frac{1}{2}g^{ab}(\bar{D}_a\bar{\Psi}D_b\Psi + D_a\Psi\bar{D}_b\bar{\Psi}) - \bar{\Psi}\Psi - V(\bar{\Psi}\Psi) - \frac{1}{4}F_{ab}F^{ab}, \quad (72)$$

where

$$F_{ab} = \partial_b\mathcal{A}_a - \partial_a\mathcal{A}_b \quad (73)$$

is the Faraday tensor with \mathcal{A}_a the vector potential,

$$D_a := \partial_a + ie\mathcal{A}_a \quad (74)$$

is a covariant derivative operator, V is the boson self-interaction term (which we assume has no explicit time dependence but is otherwise left arbitrary) and the over bar denotes complex conjugation. We decompose the spacetime into spacelike slices Σ orthogonal to a timelike Killing vector field ξ^a and write the line element as

$$ds^2 = -B(x^i)dt^2 + h_{ij}(x^i)dx^i dx^j \quad (75)$$

where i, j label spatial indices, x^i are coordinates on Σ and ξ^a has components $\xi^a = (1, 0, 0, 0)$. One can choose a gauge in which

$$\mathcal{A}_a = (\mathcal{A}_0(x^i), 0, 0, 0) \quad (76)$$

and one can show that, for the Lagrangian (72), the conserved particle number may be written as

$$N = \int_{\Sigma} \sqrt{\frac{h}{B}} [i(\Psi \partial_t \bar{\Psi} - \bar{\Psi} \partial_t \Psi) + 2e\mathcal{A}_0 \bar{\Psi} \Psi] d^3x \quad (77)$$

where the subscript ‘0’ denotes the time component and $h = \text{Det}(h_{ij})$.

The boson field Ψ has generalised momentum and velocity

$$p = \sqrt{-g} \frac{\partial L_m}{\partial(\partial_t \Psi)}, \quad \partial_t q = \partial_t \Psi \quad (78)$$

while the momentum and velocity associated with $\bar{\Psi}$ are \bar{p} and $\bar{\partial}_t q$. Since we are assuming that the spacetime is static, the Hamiltonian H for the system is given by

$$H = \int_{\Sigma} d^3x (p \partial_t q + \bar{p} \bar{\partial}_t q) - \frac{\partial I}{\partial t} + H_s \quad (79)$$

where I is given by eqn (4) using the matter Lagrangian (72) and H_s is a surface term that is independent of Ψ and $\bar{\Psi}$. In an asymptotically flat spacetime, $M_T = H \Phi_{\infty}^{-1}$.

We consider a sequence of solutions to the field equations (5), with T_{ab} derived from the Lagrangian (72), in which the fields g_{ab} , Φ , Ψ and $\bar{\Psi}$ are held fixed while $\partial_t \bar{\Psi}$ and $\partial_t \Psi$ are allowed to vary. Equivalently, we vary p and \bar{p} while keeping the conjugate variables q and \bar{q} and the gravitational fields fixed. The desired form of these time derivatives is then the one that minimises the Hamiltonian, subject to the constraint that N is conserved. Formally, we have

$$\delta(H - \Omega N) = 0 \quad (80)$$

under variations $\delta(\partial_t \Psi)$ and $\delta(\partial_t \bar{\Psi})$, where Ω is a Lagrange multiplier. Since all other fields are being held fixed, we only need to consider terms in H and N that involve time derivatives of the boson fields. These terms are

$$H_{\Psi} = \int_{\Sigma} d^3x \sqrt{\frac{h}{B}} [\partial_t \bar{\Psi} (\partial_t \Psi + ie\mathcal{A}_0 \Psi) + \partial_t \Psi (\partial_t \bar{\Psi} - ie\mathcal{A}_0 \bar{\Psi}) - \partial_t \bar{\Psi} \partial_t \Psi - ie\mathcal{A}_0 (\Psi \partial_t \bar{\Psi} - \bar{\Psi} \partial_t \Psi)] \quad (81)$$

$$= \int_{\Sigma} d^3x \sqrt{\frac{h}{B}} \partial_t \Psi \partial_t \bar{\Psi} \quad (82)$$

and

$$N_{\Psi} = \int_{\Sigma} d^3x \sqrt{\frac{h}{B}} i(\Psi \partial_t \bar{\Psi} - \bar{\Psi} \partial_t \Psi), \quad (83)$$

while eqn (80) is equivalent to

$$\delta(H_{\Psi} - \Omega N_{\Psi}) = 0. \quad (84)$$

Substituting in the explicit forms of H_{Ψ} and N_{Ψ} , eqn (84) becomes

$$\int_{\Sigma} d^3x \sqrt{\frac{h}{B}} [\delta(\partial_t \Psi) (\partial_t \bar{\Psi} + i\Omega \bar{\Psi}) + \delta(\partial_t \bar{\Psi}) (\partial_t \Psi - i\Omega \Psi)] = 0 \quad (85)$$

which can only be satisfied if the equations

$$\partial_t \Psi - i\Omega \Psi = 0, \quad \partial_t \bar{\Psi} + i\Omega \bar{\Psi} = 0 \quad (86)$$

hold. This implies that

$$\Psi = f(x^i) \exp(\Omega t) \quad (87)$$

where f is a real function. This result holds for arbitrary V (provided V has no explicit time dependence) and is independent of the form of the gravitational sector of the Lagrangian. We note here that one can also derive this result by including the variations $\delta\Psi$, $\delta(\partial_i\Psi)$ and their complex conjugates by assuming the wave equations for the boson fields are satisfied.

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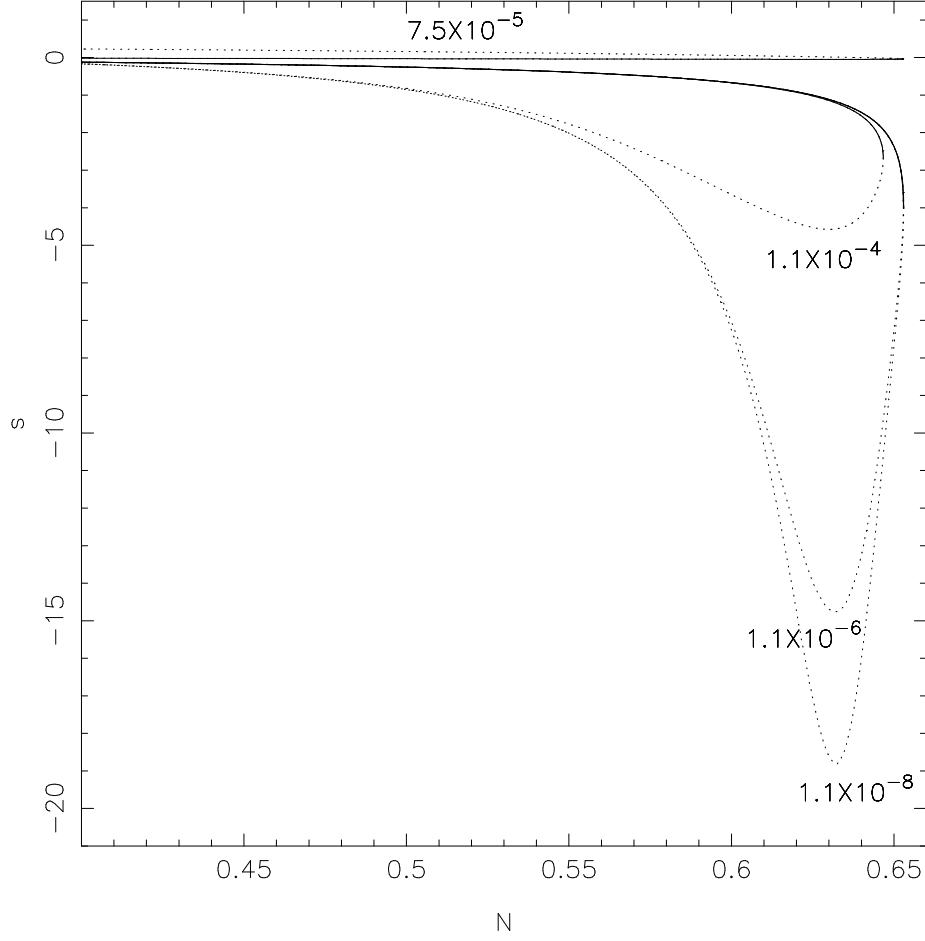


Figure 1: Sensitivity s against particle number N for ST boson stars with $2\omega + 3 = 1/(2k \log \Phi)$ and $\Lambda = 0$. The curves are labelled by the value of $(\Phi_\infty - 1)$ and the solid portion of each represents stable solutions. The uppermost curve corresponds to the parameter choice $k = 1$. The remaining three curves show solutions with $k = 3$ and, at the level of detail resolvable from the Figure, solutions with $k = 3$ and $(\Phi_\infty - 1) < 1.1 \times 10^{-8}$ are indistinguishable from the lower most curve shown.

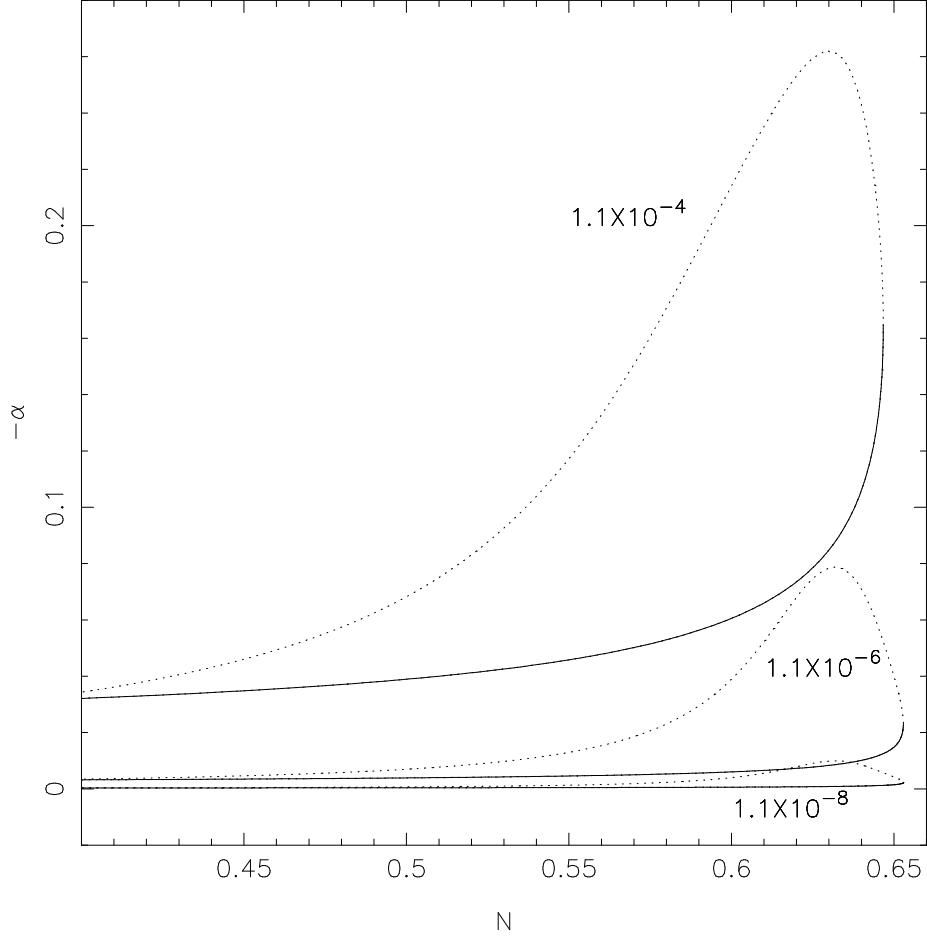


Figure 2: Coupling parameter α against particle number N for ST boson stars with $2\omega + 3 = 1/(2k \log \Phi)$ and $\Lambda = 0$. The curves are labelled by the value of $(\Phi_\infty - 1)$ and all are for $k = 3$. The solid portion of each curve represents stable solutions. As $(\Phi_\infty - 1) \rightarrow 0$, $-\alpha \rightarrow 0$ for all solutions.

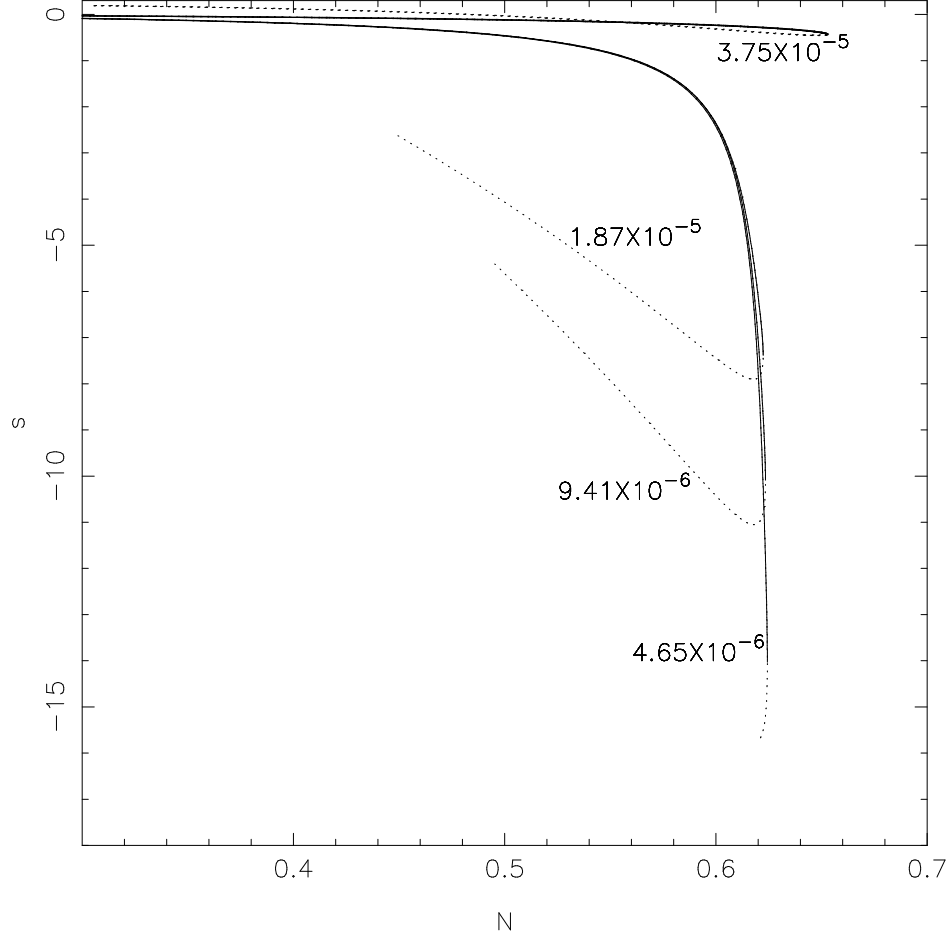


Figure 3: Sensitivity s against particle number N for ST boson stars with $\omega = \Phi/(8\xi(\Phi - 1))$ and $\Lambda = 0$. Curves are labelled by the value of $(\Phi_\infty - 1)$ and the solid portion of each represents stable solutions. The uppermost curve corresponds to the parameter choice $\xi = 1$ and is indistinguishable from $\xi = 1$ curves with $(\Phi_\infty - 1) < 3.75 \times 10^{-5}$ at the level of resolution in the Figure. The remaining three curves are for the parameter choice $\xi = 2$.

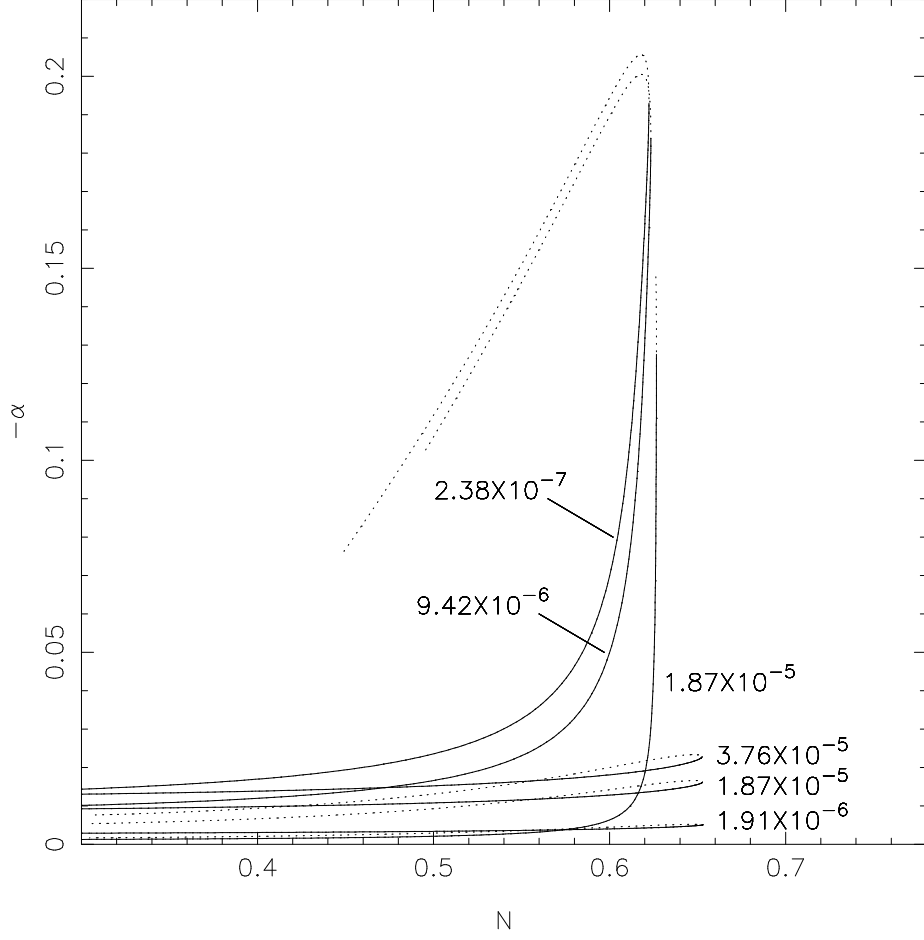


Figure 4: Coupling parameter α against particle number N for ST boson stars with $\omega = \Phi/(8\xi(\Phi - 1))$ and $\Lambda = 0$. The curves are labelled by the value of $(\Phi_\infty - 1)$ and the solid portion of each represent stable solutions. The lower three curves are for the parameter choice $\xi = 1$ while the remaining three are for $\xi = 2$.

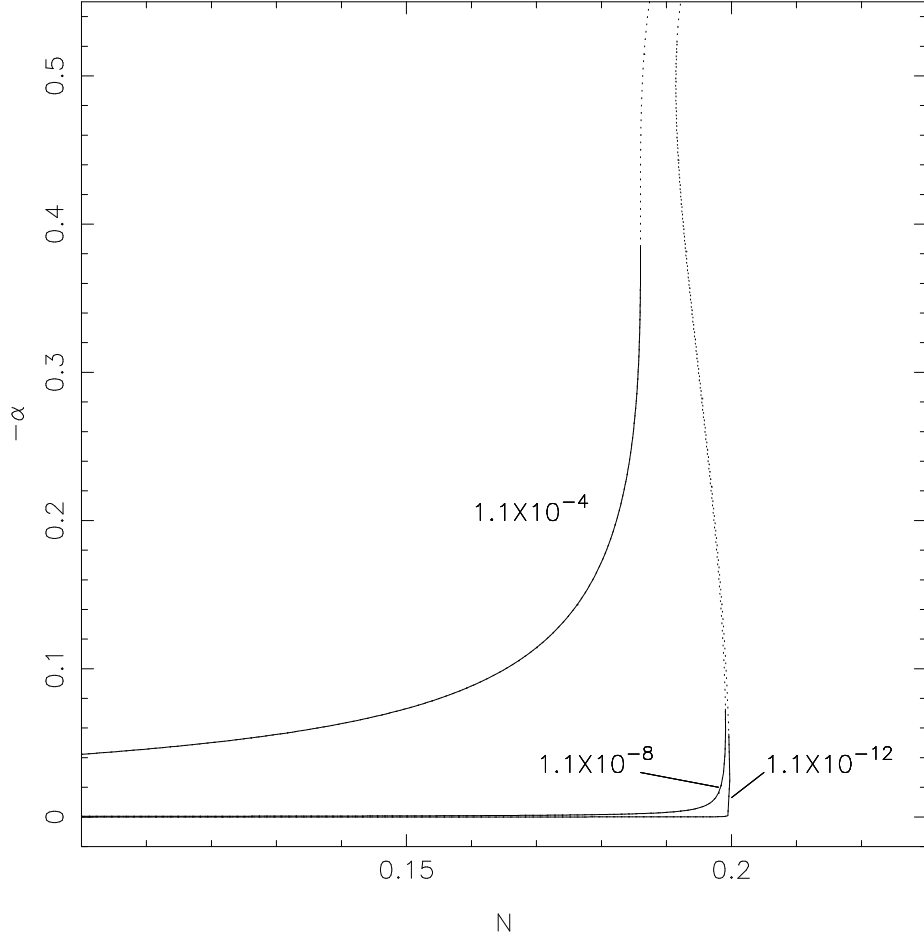


Figure 5: Coupling parameter α against particle number N for boson stars with $2\omega + 3 = 1/(2k \log \Phi)$ and in the limit $\Lambda \rightarrow \infty$. All curves are for $k = 3$ and are labelled by the value of $(\Phi_\infty - 1)$. The solid portion of each curve represents stable solutions.